IST Mathematics Core Course

Spring 2019

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1 Goals and Premises

- The idea of the course is to convey a broad view of mathematics: basic notions, problems, and results from a variety of mathematical areas, and connections between different areas.
- Apart from learning new mathematics, the goals are to practice how to familiarize yourself with concepts and questions not directly related to your main research focus, how to present and explain new material to others, how to find the right level of detail, and how to design exercises, and to encourage interaction.

2 Course Format

- To start with, you will be asked to group up in *teams of two* and choose a topic from a list we will prepare in advance; for some sample topics from previous years, see the list below.
- The task and project of each team is to prepare, with our support, four lectures and two recitations (two lectures and one recitation per team member) on your topic (this will cover two weeks of the course) . The goal is to explain and teach the main ideas of a topic to the other students as clearly as possible. This involves
  - getting an overview of the topic, possibly including finding additional references;
  - selecting which aspects and parts of the material to present in the lectures (there will not be enough time to cover everything); this may include finding good examples or focusing on illustrative special cases that convey the main ideas and difficulties of a problem or result;
  - carefully preparing a well-structured and detailed plan for the lectures (detailed notes, plus possibly some handouts, for blackboard talks, or slides for a slide talk, or a mixture of the two)
  - choosing or designing exercises for the other students to solve, and co-moderating the discussions during the recitation.

3 Timeline and Formalities

- Commit to taking the course, team up, and select your topic: We plan to hold an orientation meeting in December to discuss our suggestions of topics for this academic year. We ask that students team up and commit to taking the course soon afterwards, but not later than *January 15, 2019*;
- At least 2-3 mettings with us before the lectures, and a feed-back meeting afterwards:
  - a first "start-up" meeting 4-6 weeks before the lectures for discussing the topic, some background, and ideas which aspects could be selected for the presentation;
  - a final preparation meeting 1-2 weeks before the lectures; by this time, you should have a detailed plan of the lecture and the exercises, including notes and/or slides.
- Your grade will be based on

– Your presentation (and the effortput in during the preparation): 50%
– Homework (solving exercises): 30%
– Active participation in class: 20%

4 Example Topics

The list of topics changes every year; for this year, it will be finalized in December, before the first orientation meeting. To give you an idea of the range of topics, below is a list of some of the topics from previous years.

1. *Low-distortion embeddings of metric spaces*

   - Nathan Linial, Finite metric-spaces—combinatorics, geometry and algorithms, Proceedings of the ICM, Vol. III (Beijing, 2002), 573–586, `https://arxiv.org/abs/math/0304466v1`
   - Jiří Matoušek, Lecture notes on metric embeddings, `https://kam.mff.cuni.cz/~matousek/ba-a4.pdf`
   - Mikhail Ostrovskii, Metric embeddings. De Gruyter Studies in Mathematics, 49. De Gruyter, Berlin, 2013

Topics from Previous Years

1. *The surprising mathematics of the longest increasing subsequence*

   Pick a permutation of the numbers $\{1, \ldots, n\}$ uniformly at random. What is the expected length $\ell(n)$ of the longest increasing subsequence? This question is easy to state but not at all easy to answer. It turns out that $\ell(n) \approx 2n^{1/2} + cn^{1/6} + o(n^{1/6})$.

   The goal of this project is to present some of the key ideas in the proof of this result, which involves a surprising combination of combinatorics, analysis, probability and representation theory.

   Literature:

   - Dan Romik, The surprising mathematics of longest increasing subsequences. Institute of Mathematical Statistics Textbooks. Cambridge University Press, New York, 2015. xi+353 pp. `https://www.math.ucdavis.edu/ romik/download-book.php`

2. *The Green–Tao Theorem: Arithmetic Progressions in Primes*

   A *k-term arithmetic progression* is a set of k integers[1] of the form $a, a+d, a+2d, \ldots, a+(k-1)d$ for some $a, d \in \mathbb{Z}$, $d \neq 0$, $k \in \mathbb{N}$. Arithmetic progressions are an classical notion in number theory, with many connections to combinatorics and theoretical computer science.

   In 2004, Green and Tao, building on earlier work by Szemerédi, Gowers, Goldston, Pintz, Yıldırım, and others, proved the following long-standing conjecture (dating back to the 18th century):

   The primes contain arbitrarily long arithmetic progressions; i.e., for every $k \in \mathbb{N}$, there is an arithmetic progression of length k all of whose elements are prime numbers.

   The proof is a beautiful combination of techniques and results from both number theory and combinatorics. The goal of the project is to give an overview of the different ingredients and the structure of the proof, and then to present one or two of the main steps in more detail.

   Literature:

   - B. Green, T. Tao. Linear equations in primes. Annals of Mathematics 171 (2010), no. 3, 1753–1850.
   - D. Conlon, J. Fox, Y. Zhao. The Green-Tao theorem: an exposition. EMS Surveys in Mathematical Sciences, 1 (2014), no. 2, 249–282. `http://arxiv.org/abs/1403.2957`

[1] More generally, one can define an analogous notion for subsets of an arbitrary abelian group A instead of $\mathbb{Z}$.

3. *Gromov's Systolic Estimate*

The *systole* of a Riemannian manifold (M, g) is defined to be the smallest length of a non-contractible curve in M.

A basic question is which restrictions the topology of M imposes on the length of the systoles (up to scaling). One of the earliest results of this type is *Loewner's systolic inequality* from 1949:

If (T^2, g) is a 2-dimensional torus with a Riemannian metric g, then there is a non-contractible curve γ in T^2 whose length (with respect to g) satisfies $\text{length}(\gamma) \leq C \cdot \text{area}(T^2, g)^{1/2}$ *for some constant$C$.*

In 1983, this Gromov generalized this result from the 2-dimensional torus T^2 to higher-dimensional Tori $T^n = S^1 \times \ldots \times S^1$ (n factors) and, more generally, to *aspherical* Riemannian manifolds. Here, by definition, a topological space M is called *aspherical* if for every $k \geq 2$, every continuous map $S^k \to M$ can be continuously contracted to a point (i.e., "M contains no higher-dimensional bubbles"). Gromov's theorem asserts:

If (M, g) is an n-dimensional closed aspherical Riemannian manifold, then the systole of (M, g) is bounded from above by

$$\text{systole}(M, g) \leq C(n) \cdot \text{volume}(M, g)^{1/n},$$

where $C(n)$ is a constant only depending on n.

The goal of the project is to understand and explain the main tools and concepts in Gromov's proof, including classical ideas relating to *isoperimetric inequalities.*

Literature:[2]

- Guth, L. Notes on Gromov's systolic estimate. Geom. Dedicata, 123 (2006), 113–129.
- Guth, L. Metaphors in systolic geometry. Proceedings of the International Congress of Mathematicians. Volume II, New Delhi: Hindustan Book Agency, pp. 745–768. `http://arxiv.org/abs/1003.4247v1`

4. *Sphere Packing (in 8 and 24 dimensions)*

- Henry Cohn, A conceptual breakthrough in sphere packing. Notices Amer. Math. Soc. 64 (2017), no. 2, 102–115.
- Maryna Viazovska, The sphere packing problem in dimension 8. Ann. of Math. (2) 185 (2017), no. 3, 991–1015.
- Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, Maryna Viazovska, The sphere packing problem in dimension 24. Ann. of Math. (2) 185 (2017), no. 3, 1017?1033.

5. *The cut-off phenomenon for Markov chains*

How many times to you have to shuffle to randomise a deck of 52 cards? For a common shuffling technique (the riffle-shuffle) it turns out that 6 shuffles do usually not suffice, but the cards become very abruptly well-mixed after 7 shuffles. This is an example of the cut-off phenomenon in Markov chains, which has been intensively studied in recent years. The goal of this project is to explore the presence of a cut-off phenomenon in some interesting Markov chains.

Literature:

- Levin, David A.; Peres, Yuval; Wilmer, Elizabeth L. Markov chains and mixing times. With a chapter by James G. Propp and David B. Wilson. American Mathematical Society, Providence, RI, 2009. xviii+371 pp.

[2] Gromov's original paper *Filling Riemannian manifolds*, Journal of Differential Geometry, 18 (1983), vol. 1, 1–147 is rather long and hard to read.